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Wavelet-based integral representation for solutions of the wave equation

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Abstract

An integral representation of solutions of the wave equation as a superposition of other solutions of this equation is built. The solutions from a wide class can be used as building blocks for the representation. Considerations are based on mathematical techniques of continuous wavelet analysis. The formulae obtained are justified from the point of view of distribution theory. A comparison of the results with those by G Kaiser is carried out. Methods of obtaining physical wavelets are discussed.

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1. Introduction

The aim of this paper is to find a new exact integral representation of solutions of the wave equation. We consider a homogeneous equation with constant coefficients in a three-dimensional space. We represent a solution as a superposition of localized elementary solutions of the wave equation, which can be taken from a wide class.

A well-known exact integral representation for such a kind of wave equation is the Fourier integral, where solutions are decomposed into the superposition of plane waves. However, it is sometimes convenient to have elementary solutions well localized in the space. This is useful for studying local properties of solutions and, among other things, for studying the propagation of singularities. An approximate integral representation of solutions as a superposition of localized Gaussian beams is developed in [1–5]. However, it is heuristic and inexact.

In our paper, we use mathematical techniques of wavelet analysis. This was developed during the 1980s. The main ideas of wavelet analysis take their origin in group representation theory and in the theory of coherent states (see [6–8] and references therein). The first

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papers on continuous wavelet analysis theory were motivated by applications to seismic wave propagation [9, 10]. Nowadays a great many books and articles on wavelet analysis are available (see, for instance, [11–17]).

A special case of continuous wavelet analysis based on the analytic signal transform in a three-dimensional space was first applied to the wave equation by Kaiser in [17] and developed in [18–20]. He obtained an integral representation formula for solutions of the homogeneous wave equation as a superposition of elementary solutions derived from one fixed mother wavelet, named by him the '*physical wavelet*'. This wavelet was rigidly connected with the analytic signal transform and has power-law decrease at infinity. We give a representation in terms of solutions from a wider class.

In outline, the content of this paper is as follows. We develop the wavelet technique to construct an integral representation for solutions in section 2. We split the whole space of solutions into a direct sum of two subspaces containing solutions with positive and negative frequencies. In each of them, we choose one localized solution which must satisfy an admissibility condition. By means of this solution we construct a family of solutions by applying translations and rotations to spatial coordinates and dilations to spatial coordinates and time. The wavelet transform of solutions in each subspace does not depend on time. We use the wavelet transform of the solution in each subspace as coefficients for its integral representation. We obtain the integral formula for the solution expressed in terms of the wavelet transform of the initial data. This formula was presented for the first time in [21]. Then we give a justification of the results from the point of view of the theory of distributions. A detailed comparison with the results of Kaiser has been carried out in [22], and a brief review of it is given here. In section 3, we discuss the possibility of obtaining new physical wavelets by means of some known methods of constructing explicit exact solutions of the wave equation. We also consider exponentially localized physical wavelets which were found and generalized in [23–25], the wavelet properties of which have been studied in [26].

2. Integral representation for solutions of the wave equation

We seek an integral representation of solutions of the wave equation in the form

$$u(\boldsymbol{r}, t) = \int \mathrm{d}\mu(\nu) U(\nu) \varphi^{\nu}(\boldsymbol{r}, t), \tag{1}$$

where ν is a set of parameters, $\int \mathrm{d}\mu(\nu)$ denotes integration with respect to the measure $\mu(\nu)$ in the space of parameters, $\varphi^{\nu}(\boldsymbol{r}, t)$ is a family of elementary solutions dependent on the parameter ν, and $U(\nu)$ are coefficients. In the following sections, we define each of these objects.

2.1. The space $\mathcal{H}$ of solutions of the wave equation

Consider the homogeneous wave equation in $\mathbb{R}^3$ with a constant coefficient c:

$$\Box u \equiv u_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) = 0, \qquad \boldsymbol{r} = (x, y, z). \tag{2}$$

We fix the space $\mathcal{H}$ of complex-valued solutions $u(\boldsymbol{r}, t)$ of the wave equation, which are square integrable with respect to the spatial coordinate $\boldsymbol{r}$ when time t is fixed. If the function $u(\boldsymbol{r}, t)$ is not smooth enough, it is a solution understood in the sense of distributions (see section 2.5).

The Fourier transform of the solution u taken with respect to coordinates $\boldsymbol{r}$ reads

$$\widehat{u}(\boldsymbol{k}, t) = \widehat{u}_{+}(\boldsymbol{k}, 0)\,\mathrm{e}^{-\mathrm{i}|\boldsymbol{k}|ct} + \widehat{u}_{-}(\boldsymbol{k}, 0)\,\mathrm{e}^{\mathrm{i}|\boldsymbol{k}|ct}, \qquad \widehat{u}_{\pm}(\boldsymbol{k}, 0) \in L_2(\mathbb{R}^3). \tag{3}$$

The space of solutions $\mathcal{H}$ is decomposed into a direct sum of two subspaces $\mathcal{H}_\pm$ of positive and negative frequencies:

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-, \qquad u(\boldsymbol{r}, t) = u_+(\boldsymbol{r}, t) + u_-(\boldsymbol{r}, t). \tag{4}$$

In the space $\mathcal{H}_+$, we introduce a standard $L_2(\mathbb{R}^3)$ scalar product with respect to the spatial coordinates $\boldsymbol{r}$:

$$\langle u_+, v_+\rangle \equiv \int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{r}\, u_+(\boldsymbol{r}, t)\overline{v_+(\boldsymbol{r}, t)} = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{k}\, \widehat{u}_+(\boldsymbol{k}, 0)\overline{\widehat{v}_+(\boldsymbol{k}, 0)}. \tag{5}$$

This scalar product does not depend on time t. This is the main reason why we decompose the whole $\mathcal{H}$ into a direct sum of these two subspaces $\mathcal{H}_\pm$. If we try to use the standard $L_2(\mathbb{R}^3)$ scalar product directly in the space $\mathcal{H}$, the exponents $\exp(\mathrm{i}|\boldsymbol{k}|ct)$ and $\exp(-\mathrm{i}|\boldsymbol{k}|ct)$ do not cancel, and the time dependance is not removed.

2.2. *A family of elementary solutions*

We fix a solution $\psi_+(\boldsymbol{r}, t) \in \mathcal{H}_+$ satisfying the admissibility condition:

$$C_\psi^+ \equiv \int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{k} \frac{|\widehat{\psi}_+(\boldsymbol{k}, 0)|^2}{|\boldsymbol{k}|^3} < \infty. \tag{6}$$

For the case $\psi_+(\boldsymbol{r}, 0) \in L_1(\mathbb{R}^3)$, the condition (6) holds if $\widehat{\psi}_+(\mathbf{o}, 0) = 0$. This solution is named a 'physical wavelet' in accordance with the Kaiser terminology [17].

We construct a family of elementary solutions $\psi_+^\nu(\boldsymbol{r}, t)$ in the following way:

$$\psi_+^\nu(\boldsymbol{r}, t) \equiv \frac{1}{a^{3/2}}\psi_+\left(\mathrm{M}^{-1}_{\vartheta_1\vartheta_2\vartheta_3}\frac{\boldsymbol{r}-\boldsymbol{b}}{a}, \frac{t}{a}\right), \qquad \nu = (a, \boldsymbol{b}, \vartheta_1, \vartheta_2, \vartheta_3). \tag{7}$$

The rotation matrix M depends on Euler angles as follows:

$$\begin{aligned} M_{\vartheta_1\vartheta_2\vartheta_3} &= M_{\vartheta_1}M_{\vartheta_2}M_{\vartheta_3} \\ &= \begin{pmatrix} \cos\vartheta_1 & -\sin\vartheta_1 & 0 \\ \sin\vartheta_1 & \cos\vartheta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta_2 & -\sin\vartheta_2 \\ 0 & \sin\vartheta_2 & \cos\vartheta_2 \end{pmatrix} \begin{pmatrix} \cos\vartheta_3 & -\sin\vartheta_3 & 0 \\ \sin\vartheta_3 & \cos\vartheta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{8}$$

If ψ_+ has a symmetry with respect to the OZ axis, the product $M^{-1}_{\vartheta_1\vartheta_2\vartheta_3}$ can be replaced by $M^{-1}_{\vartheta_2}M^{-1}_{\vartheta_1}$. Then the three parameters $a, \vartheta_1, \vartheta_2$ have the meaning of a spatial frequency vector $\boldsymbol{q}$, where $|\boldsymbol{q}| = 1/a$ and the angles determine its direction in the spherical coordinate system (see [12]).

The family of solutions (7) is a special case of space-temporal wavelets [12] but without temporal translations and a speed tuning transformation.

We define the wavelet transform of the solution $u_+(\boldsymbol{r}, t)$ as

$$\begin{aligned} U_+(\nu) \equiv \langle u_+, \psi_+^\nu\rangle &= \frac{1}{a^{3/2}}\int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{r}\, u_+(\boldsymbol{r}, t)\overline{\psi_+\left(M^{-1}_{\vartheta_1,\vartheta_2,\vartheta_3}\frac{\boldsymbol{r}-\boldsymbol{b}}{a}, \frac{t}{a}\right)} \\ &= \frac{a^{3/2}}{(2\pi)^3}\int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{k}\, \widehat{u}_+(\boldsymbol{r}, 0)\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{b})\overline{\widehat{\psi}_+\big(aM^{-1}_{\vartheta_1,\vartheta_2,\vartheta_3}\boldsymbol{k}, 0\big)}. \end{aligned} \tag{9}$$

The distinction between this version of the transform and the space-temporal transform in [12] is in the absence of integration over time in (9). However $U_+(\nu)$ do not depend on time because of the choice of the scalar product in $\mathcal{H}_+$.

2.3. Isometry property and integral representation

The isometry property of the wavelet transform of square integrable functions $u_+(\boldsymbol{r}, 0)$, $v_+(\boldsymbol{r}, 0)$ yields [11, 12]

$$\langle u_+, v_+\rangle = \frac{1}{C_\psi^+}\int \mathrm{d}\mu(\nu) U_+(\nu)\overline{V_+(\nu)}, \qquad \forall u_+, v_+ \in \mathcal{H}_+, \tag{10}$$

$$\int \mathrm{d}\mu(\nu) = \int_0^{2\pi} \mathrm{d}\vartheta_1 \int_0^{\pi} \mathrm{d}\vartheta_2 \sin\vartheta_2 \int_0^{2\pi} \mathrm{d}\vartheta_3 \int_0^{\infty} \frac{\mathrm{d}a}{a^4} \int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{b}. \tag{11}$$

This property extends to any other moment of time, because neither $U_+(\nu)$, $V_+(\nu)$ nor the scalar product of u_+ and v_+ depend on time t. The isometry property implies the reconstruction formula

$$u_+(\boldsymbol{r}, t) = \frac{1}{C_\psi^+}\int \mathrm{d}\mu(\nu) U_+(\nu)\psi_+^\nu(\boldsymbol{r}, t), \tag{12}$$

which holds in the weak sense for any $u_+ \in \mathcal{H}_+$. The coefficients $U_+(\nu)$ do not depend on $\boldsymbol{r}$ and t and thus the formula (12) has the meaning of a superposition of elementary solutions $\psi_+^\nu(\boldsymbol{r}, t)$.

The decomposition (12) is also valid in the case where $U_+ = \langle u_+, \chi_+^\nu\rangle$, $\chi_+(\boldsymbol{r}, t) \in \mathcal{H}_+$ owing to [11, 12]. Then

$$C_{\psi\chi}^+ \equiv \int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{k} \frac{\widehat{\psi}_+(\boldsymbol{k}, 0)\overline{\widehat{\chi}_+(\boldsymbol{k}, 0)}}{|\boldsymbol{k}|^3} \tag{13}$$

stands for C_ψ^+. This means that the definition of coefficients for one and the same solution is not unique.

The case of $\mathcal{H}_-$ is considered in a similar way and the whole representation reads

$$u(\boldsymbol{r}, t) = \frac{1}{C_\psi^+}\int \mathrm{d}\mu(\nu) U_+(\nu)\psi_+^\nu(\boldsymbol{r}, t) + \frac{1}{C_\psi^-}\int \mathrm{d}\mu(\nu) U_-(\nu)\psi_-^\nu(\boldsymbol{r}, t). \tag{14}$$

2.4. Initial-value problem for the wave equation

Each solution of the wave equation from $\mathcal{H}_\pm$ can easily be expressed in terms of its initial-value problem. Consider the following initial-value problem:

$$\begin{cases} u_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) = 0, \\ u|_{t=0} = w(\boldsymbol{r}), \qquad \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = v(\boldsymbol{r}), \end{cases} \tag{15}$$
$$w(\boldsymbol{r}) \in L_2(\mathbb{R}^3), \widehat{v}(\boldsymbol{k})/|\boldsymbol{k}| \in L_2(\mathbb{R}^3).$$

We seek the solution of the form (14). To find the coefficient $U_\pm(\nu)$, we must split the solution u into u_+ and u_-. We obtain $\widehat{u}_\pm(\boldsymbol{k}, 0)$ from the initial-value data:

$$\widehat{u}_+(\boldsymbol{k}, 0) = \frac{1}{2}\left[\widehat{w}(\boldsymbol{k}) - \frac{1}{\mathrm{i}c|\boldsymbol{k}|}\widehat{v}(\boldsymbol{k})\right], \qquad \widehat{u}_-(\boldsymbol{k}, 0) = \frac{1}{2}\left[\widehat{w}(\boldsymbol{k}) + \frac{1}{\mathrm{i}c|\boldsymbol{k}|}\widehat{v}(\boldsymbol{k})\right]. \tag{16}$$

Substituting $\widehat{u}_\pm$ from (16) into (9), we obtain

$$U_\pm(\nu) = \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{k}\, \widehat{w}(\boldsymbol{k})\overline{\widehat{\psi}_\pm^\nu(\boldsymbol{k}, 0)} \mp \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{k} \frac{1}{\mathrm{i}c|\boldsymbol{k}|}\widehat{v}(\boldsymbol{k})\overline{\widehat{\psi}_\pm^\nu(\boldsymbol{k}, 0)}. \tag{17}$$

The Plancherel equality yields

$$\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{k}\,\widehat{w}(\boldsymbol{k})\overline{\widehat{\psi}^{\nu}_{\pm}(\boldsymbol{k},0)} = \langle w(\boldsymbol{r}), \psi^{\nu}_{\pm}(\boldsymbol{r},0)\rangle \equiv W_{\pm}(\nu), \tag{18}$$

$$\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \mathrm{d}^3\boldsymbol{k}\frac{1}{\mathrm{i}c|\boldsymbol{k}|}\widehat{v}(\boldsymbol{k})\overline{\widehat{\psi}^{\nu}_{\pm}(\boldsymbol{k},0)} = a\langle v(\boldsymbol{r}), \chi^{\nu}_{\pm}(\boldsymbol{r},0)\rangle \equiv aV_{\pm}(\nu), \tag{19}$$

where

$$\widehat{\chi}^{\nu}_{\pm}(\boldsymbol{k},0) = -\frac{\widehat{\psi}^{\nu}_{\pm}(\boldsymbol{k},0)}{\mathrm{i}ac|\boldsymbol{k}|}. \tag{20}$$

Finally, the integral representation of $u(\boldsymbol{r},t)$ reads

$$u(\boldsymbol{r},t) = \frac{1}{C^+_\psi}\int \mathrm{d}\mu(\nu)\left[\frac{1}{2}W_+(\nu) - \frac{a}{2}V_+(\nu)\right]\psi^{\nu}_{+}(\boldsymbol{r},t) + \frac{1}{C^-_\psi}\int \mathrm{d}\mu(\nu)\left[\frac{1}{2}W_-(\nu) + \frac{a}{2}V_-(\nu)\right]\psi^{\nu}_{-}(\boldsymbol{r},t). \tag{21}$$

2.5. The result in the sense of distributions

The justification of the decomposition (21) requires some discussion of the convergence problems. Instead of this, we show that (21) can be understood in the sense of distributions [18]. We construct test functions, employing infinitely differentiable solutions $\beta_{\pm}(\boldsymbol{r},t)$, $\beta_-(\boldsymbol{r},t) = \beta_+(\boldsymbol{r},-t)$. We define test functions $\beta(\boldsymbol{r})$ as $\beta(\boldsymbol{r}) = \beta_+(\boldsymbol{r},0) = \beta_-(\boldsymbol{r},0)$. We assume that $\beta(\boldsymbol{r})$ and their derivatives decrease with $|\boldsymbol{r}|$ faster than $|\boldsymbol{r}|^{-n}$ for any n. Formula (21) follows from (12) and its analog for negative-frequency solutions. It is understood in the sense of distributions if the following relation

$$\langle u_+(\boldsymbol{r},t), \beta(\boldsymbol{r})\rangle = \frac{1}{C^+_\psi}\int \mathrm{d}\mu(\nu)U_+(\nu)\big\langle\psi^{\nu}_{+}(\boldsymbol{r},t), \beta(\boldsymbol{r})\big\rangle \tag{22}$$

is valid for any test function $\beta(\boldsymbol{r})$. To prove this relation, we consider scalar products in it in the spatial frequency domain and associate the time-dependent exponent with $\widehat{\beta}(\boldsymbol{k})$. Then we obtain

$$\langle u_+(\boldsymbol{r},t), \beta(\boldsymbol{r})\rangle = \langle u_+(\boldsymbol{r},0), \beta_-(\boldsymbol{r},t)\rangle, \qquad \big\langle\psi^{\nu}_{+}(\boldsymbol{r},t), \beta(\boldsymbol{r})\big\rangle = \big\langle\psi^{\nu}_{+}(\boldsymbol{r},0), \beta_-(\boldsymbol{r},t)\big\rangle. \tag{23}$$

The isometry property for the continuous wavelet transform yields

$$\langle u_+(\boldsymbol{r},0), \beta_-(\boldsymbol{r},t)\rangle = \frac{1}{C^+_\psi}\int \mathrm{d}\mu(\nu)U_+(\nu)\overline{B_-(\nu,t)}, \tag{24}$$

where $B_-(\nu,t)$ is the wavelet transform of the function $\beta_-(\boldsymbol{r},t)$:

$$B_-(\nu,t) = \langle \beta_-(\boldsymbol{r},t), \psi^{\nu}_{+}(\boldsymbol{r},0)\rangle, \tag{25}$$

time t being a parameter. Formula (22) follows from (24), (23) and (25).

The functions defined by (3) may have discontinuities and thus they do not in general represent classical solutions of (2). A square integrable function $u(\boldsymbol{r},t)$ is a solution of the initial-value problem in the sense of distribution if it satisfies the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle u(\boldsymbol{r},t), \beta(\boldsymbol{r})\rangle = c^2\langle u(\boldsymbol{r},t), \triangle\beta(\boldsymbol{r})\rangle, \tag{26}$$

and the initial conditions

$$\langle u(\boldsymbol{r},t), \beta(\boldsymbol{r})\rangle|_{t=0} = \langle w(\boldsymbol{r}), \beta(\boldsymbol{r})\rangle, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\langle u(\boldsymbol{r},t), \beta(\boldsymbol{r})\rangle\bigg|_{t=0} = \langle v(\boldsymbol{r}), \beta(\boldsymbol{r})\rangle \tag{27}$$

for any test function $\beta(\boldsymbol{r})$.

We obtain (26) for positive and negative frequencies individually. We put in (26), say, $u(\boldsymbol{r},t) = u_+(\boldsymbol{r},t)$, pass to the spatial frequency domain in scalar products and associate the time-dependent exponent with $\widehat{\beta}(\boldsymbol{k})$. Then we get

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle u_+(\boldsymbol{r},0),\beta_-(\boldsymbol{r},t)\rangle = c^2\langle u_+(\boldsymbol{r},0),\triangle\beta_-(\boldsymbol{r},t)\rangle. \tag{28}$$

It is satisfied, because $\beta_-(\boldsymbol{r},t)$ is a smooth solution of the wave equation.

Now we turn to the initial data (27). We write formula (21) in the following notation:

$$\langle u(\boldsymbol{r},t),\beta(\boldsymbol{r})\rangle = \tfrac{1}{2}(I_{W+}+I_{W-}+I_{V+}+I_{V-}), \tag{29}$$

where

$$I_{W\pm} = \frac{1}{C_\psi^\pm}\int \mathrm{d}\mu(\nu)W_\pm(\nu)\big\langle\psi_\pm^\nu(\boldsymbol{r},t),\beta(\boldsymbol{r})\big\rangle,$$

$$I_{V\pm} = \mp\frac{1}{C_\psi^\pm}\int \mathrm{d}\mu(\nu)aV_\pm(\nu)\big\langle\psi_\pm^\nu(\boldsymbol{r},t),\beta(\boldsymbol{r})\big\rangle.$$

First, we put time t equal to zero and check if the right-hand side of the formula (29) satisfies the first of the initial conditions (27). The terms I_{W+} and I_{W-} are converted to $\langle w(\boldsymbol{r}),\beta(\boldsymbol{r})\rangle$, owing to the isometry property. The term I_{V+} turns to $-\langle\widetilde{v}(\boldsymbol{r}),\beta(\boldsymbol{r})\rangle$ where

$$\widetilde{v}(\boldsymbol{r}) \equiv \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\mathrm{d}^3\boldsymbol{k}\,\frac{1}{\mathrm{i}c|\boldsymbol{k}|}\widehat{v}(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{30}$$

the term I_{V-} gives the same but with an opposite sign, and thus I_{V+} and I_{V-} cancel each other. The right-hand side of the formula (29) then reads $\langle w(\boldsymbol{r}),\beta(\boldsymbol{r})\rangle$.

Second, we take the time derivative of the expression (29) and put the time equal to zero. We note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\langle\psi_\pm^\nu(\boldsymbol{r},t),\beta(\boldsymbol{r})\big\rangle\Big|_{t=0} = \left\langle\psi_\pm^\nu(\boldsymbol{r},0),\ \frac{\mathrm{d}}{\mathrm{d}t}\beta_\mp(\boldsymbol{r},t)\Big|_{t=0}\right\rangle = \pm\big\langle\psi_\pm^\nu(\boldsymbol{r},0),\dot{\beta}(\boldsymbol{r})\big\rangle, \tag{31}$$

where

$$\dot{\beta}(\boldsymbol{r}) \equiv \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\mathrm{d}^3\boldsymbol{k}\,\mathrm{i}c|\boldsymbol{k}|\widehat{\beta}(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}, \qquad \dot{\beta}\in S(\mathbb{R}^3). \tag{32}$$

Then $\dot{I}_{W\pm} = \pm\langle w(\boldsymbol{r}),\dot{\beta}(\boldsymbol{r})\rangle$ and $\dot{I}_{W+}+\dot{I}_{W-} = 0$. The dot over I denotes a time derivative of I when $t=0$. Applying the same arguments to the term $\dot{I}_{V+}$, we obtain

$$\dot{I}_{V+} = -\langle\widetilde{v}(\boldsymbol{r}),\dot{\beta}(\boldsymbol{r})\rangle = -\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\mathrm{d}^3\boldsymbol{k}\,\frac{1}{\mathrm{i}c|\boldsymbol{k}|}\widehat{v}(\boldsymbol{k})\overline{\mathrm{i}c|\boldsymbol{k}|\widehat{\beta}(\boldsymbol{k})} = \langle v(\boldsymbol{r}),\beta(\boldsymbol{r})\rangle. \tag{33}$$

The term $\dot{I}_{V-}$ gives the same expression, whence we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u(\boldsymbol{r},t),\beta(\boldsymbol{r})\rangle\Big|_{t=0} = \langle v(\boldsymbol{r}),\beta(\boldsymbol{r})\rangle, \tag{34}$$

and the conditions (27) are satisfied.

3. Some examples of physical wavelets

As is seen from section 2.1, we can construct the physical wavelet for $\mathcal{H}_\pm$ just by choosing its Fourier transform $\widehat{\psi}_\pm(\boldsymbol{k},0)\in\mathbb{L}_2(\mathbb{R}^3)$ having a root of any order at the point $\boldsymbol{k}=0$. However, in practice, we possibly will be unable to find an analytic explicit formula for the wavelet ψ in the position space. There are methods which allow one to obtain such formulae [28–32] (see [33] for a review of such methods). The aim of this section is to look at some of these methods from the point of view of physical wavelets.

3.1. Spherically symmetric mother wavelets

Consider two solutions of the wave equation $g_{1,2}(\boldsymbol{r}, t)$:

$$g_{1,2}(\boldsymbol{r}, t) = \frac{1}{4\pi c^2}\frac{\phi(ct \mp |\boldsymbol{r}|)}{|\boldsymbol{r}|}, \qquad \left(\frac{\partial^2}{\partial t^2} - c^2\Delta\right) g_{1,2}(\boldsymbol{r}, t) = \phi(ct)\delta(\boldsymbol{r}), \tag{35}$$

where ϕ is a function of time t which was called by Kaiser a '*proxy wavelet*' [17]. The solution $g_1(\boldsymbol{r}, t)$ has the meaning of a wave emitted by a point source at the origin, and the solution $g_2(\boldsymbol{r}, t)$ represents a wave absorbed by a point source. The difference

$$\psi(\boldsymbol{r}, t) = g_2(\boldsymbol{r}, t) - g_1(\boldsymbol{r}, t) = \frac{1}{4\pi c^2|\boldsymbol{r}|}\left[\phi(ct + |\boldsymbol{r}|) - \phi(ct - |\boldsymbol{r}|)\right] \tag{36}$$

has no singularities and is a solution of the homogeneous wave equation (2). The Fourier transform of the solution $\psi(\boldsymbol{r}, t)$ (36) can be calculated exactly:

$$\widehat{\psi}(\boldsymbol{k}, t) = -\frac{\widehat{\phi}(|\boldsymbol{k}|)}{2\mathrm{i}|\boldsymbol{k}|c^2}\exp(\mathrm{i}|\boldsymbol{k}|ct) + \frac{\widehat{\phi}(-|\boldsymbol{k}|)}{2\mathrm{i}|\boldsymbol{k}|c^2}\exp(-\mathrm{i}|\boldsymbol{k}|ct). \tag{37}$$

It splits into positive-frequency and negative-frequency parts.

If we choose a progressive proxy wavelet $\phi(t)$, i.e., $\widehat{\phi}(\xi) \equiv 0$ for $\xi < 0$, then $\psi \in \mathcal{H}_-$, i.e. $\psi(\boldsymbol{r}, t) \equiv \psi_-(\boldsymbol{r}, t)$ and we can take $\psi_+(\boldsymbol{r}, t) = \psi_-(\boldsymbol{r}, -t)$. The admissibility condition (6) then can be stated in terms of the proxy wavelet ϕ in the way

$$C_\psi^+ = C_\psi^- = \frac{\pi}{c^4}\int_0^{+\infty} \mathrm{d}k \frac{|\widehat{\phi}(k)|^2}{k^3} < \infty. \tag{38}$$

The physical wavelet (A.8) by Kaiser [17] was derived from the following proxy wavelet:

$$\phi(t) = \Gamma(\alpha)(1 - \mathrm{i}t)^{-\alpha}/\pi, \qquad \widehat{\phi}(\xi) = 2\Theta(\xi)\xi^{\alpha-1}\exp(-\xi), \tag{39}$$

where Θ is the Heaviside step function. In the position space, the physical wavelet reads

$$\psi_-(\boldsymbol{r}, t) = \frac{\Gamma(\alpha)}{4\pi^2 c^2|\boldsymbol{r}|}\left[\frac{1}{[1 - \mathrm{i}(ct + |\boldsymbol{r}|)]^\alpha} - \frac{1}{[1 - \mathrm{i}(ct - |\boldsymbol{r}|)]^\alpha}\right]. \tag{40}$$

In [22], we proposed to choose

$$\phi(t) = \exp\left(-2\sqrt{1 - \mathrm{i}t}\right), \tag{41}$$

where the branch of the square root with positive real part is implied. This proxy wavelet is progressive, and it produces a physical wavelet of the form

$$\psi_-(\boldsymbol{r}, t) = \frac{1}{4\pi c^2|\boldsymbol{r}|}\left\{\exp\left[-2\sqrt{1 - \mathrm{i}(ct + |\boldsymbol{r}|)}\right] - \exp\left[-2\sqrt{1 - \mathrm{i}(ct - |\boldsymbol{r}|)}\right]\right\}. \tag{42}$$

Such a solution was mentioned in [34]. Its Fourier transform $\widehat{\psi}_-(\boldsymbol{k}, t)$ reads

$$\widehat{\psi}_-(\boldsymbol{k}, 0) = \frac{\mathrm{i}\sqrt{\pi}}{c^2}|\boldsymbol{k}|^{-5/2}\exp\left[-|\boldsymbol{k}| - \frac{1}{|\boldsymbol{k}|}\right]. \tag{43}$$

The Fourier transform of this wavelet has a root of infinite order at the origin $\boldsymbol{k} = 0$, owing to the factor $\exp(-1/|\boldsymbol{k}|)$, and the wavelet itself has an infinite number of vanishing moments. The wavelet ψ_- has a spherical symmetry and an exponential decay away from the circle $|\boldsymbol{r}| = ct$. The coefficient C_ψ^- (6) can be calculated exactly:

$$C_\psi^- = 4\pi\int_0^\infty \mathrm{d}k \frac{|\widehat{\psi}_-(k, 0)|^2}{k} = \frac{8\pi^2}{c^4}K_5(4) < \infty, \tag{44}$$

where $K_5(4)$ is the McDonald function [35].

3.2. Nonsymmetric mother wavelets

We discuss here the construction of nonsymmetric solutions of the wave equation (2) following papers [31, 32]. The method is based on the summation of nonstationary Gaussian beams [36, 37] multiplied by a weight function $\widehat{\phi}(q)$:

$$\psi(\boldsymbol{r},t) = \int_0^{+\infty} \mathrm{d}q\,\widehat{\phi}(q)\psi_{\text{beam}}(q,\boldsymbol{r},t), \tag{45}$$

$$\psi_{\text{beam}}(q,\boldsymbol{r},t) = \frac{\exp\left[\mathrm{i}q\theta(\boldsymbol{r},t)\right]}{\sqrt{x+ct-\mathrm{i}\varepsilon_1}\sqrt{x+ct-\mathrm{i}\varepsilon_2}}, \qquad q>0, \tag{46}$$

$$\theta(\boldsymbol{r},t) = x - ct + \frac{y^2}{x+ct-\mathrm{i}\varepsilon_1} + \frac{z^2}{x+ct-\mathrm{i}\varepsilon_2},$$

where ε_1 and ε_2 are free positive parameters. If $\widehat{\phi}(q)\equiv 0, q<0$, the formula (45) has the meaning of a Fourier inverse transform and can be written in a simpler form:

$$\psi(\boldsymbol{r},t) = \frac{1}{\sqrt{x+ct-\mathrm{i}\varepsilon_1}\sqrt{x+ct-\mathrm{i}\varepsilon_2}}\phi[\theta(\boldsymbol{r},t)], \tag{47}$$

where ϕ is a Fourier inverse transform of $\widehat{\phi}$. The function ϕ then can be named a 'proxy wavelet'. The formula (47) is a special case of the class of solutions presented by Bateman in [28, 29] and further developed by Hillion in [30]. Now we determine the class of proxy wavelets ϕ that produce admissible physical wavelets. The Fourier transform of a solution $\psi(\boldsymbol{r},t)$ defined by (47) reads

$$\widehat{\psi}(\boldsymbol{k},t) = 2\pi^2\,\mathrm{i}\widehat{\phi}\left(\frac{k_{\mathrm{x}}+|\boldsymbol{k}|}{2}\right)\frac{1}{|\boldsymbol{k}|}\exp\left[-\mathrm{i}|\boldsymbol{k}|ct - \frac{k_{\mathrm{y}}^2\varepsilon_1 + k_{\mathrm{z}}^2\varepsilon_2}{2(k_{\mathrm{x}}+|\boldsymbol{k}|)}\right], \qquad \boldsymbol{k}=(k_{\mathrm{x}},k_{\mathrm{y}},k_{\mathrm{z}}). \tag{48}$$

Substituting this expression into the formula for the coefficient C_ψ defined by (6), we conclude that $\widehat{\phi}(q)$ must have a root of order at least $1+\alpha, \alpha>0$, at the origin $q=0$. We also restrict the class of admissible proxy wavelets to the class $L_1(\mathbb{R})\bigcap L_2(\mathbb{R})$.

A special case of solutions of the class (47) named the Gaussian wave packet was found in [23, 24, 38] and studied in [26],

$$\psi(\boldsymbol{r},t) = \frac{1}{\sqrt{x+ct-\mathrm{i}\varepsilon_1}\sqrt{x+ct-\mathrm{i}\varepsilon_2}}\exp\left[-p\sqrt{1-\frac{\mathrm{i}\theta(\boldsymbol{r},t)}{\gamma}}\right], \tag{49}$$

where p and γ are free positive parameters. This solution can be obtained from (47) by using the following proxy wavelet:

$$\phi(t) = \exp\left[-p\sqrt{1-\frac{\mathrm{i}t}{\gamma}}\right]. \tag{50}$$

The Fourier transform of the Gaussian packet (49) due to (48) and (50) has the form

$$\widehat{\psi}(\boldsymbol{k},t) = \mathrm{i}(2\pi)^{3/2}\frac{p}{\sqrt{\gamma}}\frac{1}{|\boldsymbol{k}|(|\boldsymbol{k}|+k_{\mathrm{x}})^{3/2}}$$
$$\times\exp\left[-\frac{|\boldsymbol{k}|+k_{\mathrm{x}}}{2}\gamma - \frac{p^2}{2\gamma}\frac{1}{|\boldsymbol{k}|+k_{\mathrm{x}}} - \frac{k_{\mathrm{y}}^2\varepsilon_1+k_{\mathrm{z}}^2\varepsilon_2}{2(k_{\mathrm{x}}+|\boldsymbol{k}|)} - \mathrm{i}|\boldsymbol{k}|ct\right]. \tag{51}$$

This physical wavelet has an exponential decay away from the moving point $x=ct, y=0, z=0$. It has infinitely many vanishing moments with respect to spatial coordinates. As is shown in [26], its asymptotics coincides with the Morlet wavelet [11, 12] as $p \to \infty$ and time t is fixed:

$$\psi(\boldsymbol{r}, t) = \frac{C}{(-\mathrm{i}\varepsilon_1)^{1/2}(-\mathrm{i}\varepsilon_2)^{1/2}} \exp\left[\mathrm{i}\varkappa(x - ct) - \frac{(x-ct)^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} - \frac{z^2}{2\sigma_z^2}\right][1 + \mathrm{O}(p^{-3\alpha+1})],$$
$$\alpha \in (1/3, 1/2), \tag{52}$$

where

$$\sigma_x^2 = 4\gamma^2/p, \qquad \sigma_y^2 = \gamma\varepsilon_1/p, \qquad \sigma_z^2 = \gamma\varepsilon_2/p, \qquad \varkappa = p/(2\gamma), \tag{53}$$

in the domain

$$(x - ct)/\gamma = \mathrm{O}(p^{-\alpha}), \qquad y/\sqrt{\varepsilon_1\gamma} = \mathrm{O}(p^{-\alpha}), \qquad z/\sqrt{\varepsilon_2\gamma} = \mathrm{O}(-p^{\alpha}), \tag{54}$$

provided that the parameters $2ct/\varepsilon_j$, $p^{-\alpha}\gamma/\varepsilon_j$, $j = 1, 2$ are small.

The axially symmetric case of the Gaussian beam (46) and the Gaussian packet (49) can be obtained by putting $\varepsilon_1 = \varepsilon_2$.

4. Conclusions

We develop methods of continuous wavelet analysis to obtain a new integral representation of solutions of the wave equation in 3D space. Every solution is decomposed in terms of elementary ones. The coefficients of the decomposition are expressed in terms of the wavelet transform of the initial data. Examples of elementary solutions are given.

Appendix. The representation built by G Kaiser

G Kaiser was the first to apply continuous wavelet analysis to decompose solutions of the wave equation. He used the analytic signal transform (AST) [17] of a function $u(\boldsymbol{r}, t)$:

$$U(\boldsymbol{b}, a) = \frac{1}{\pi\mathrm{i}} \int_{\mathbb{R}} \frac{\mathrm{d}\tau}{\tau - \mathrm{i}} u(\boldsymbol{b}, \tau a), \qquad \boldsymbol{b} \in \mathbb{R}^3, a \neq 0, \tag{A.1}$$

as a base for the definition of the wavelet transform of this function. He suggests writing (A.1) as follows:

$$U(\boldsymbol{b}, a) = \langle u, \Psi^{\boldsymbol{b},a}\rangle, \tag{A.2}$$

where $\Psi^{\boldsymbol{b},a}(\boldsymbol{r}, t)$ is a four-parameter family of wavelets, which arise naturally and are called acoustic wavelets. The inner product of two solutions u, v is defined in the Fourier domain by the formula

$$\langle u, v\rangle = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{k}}{2\omega^\alpha} [\widehat{u}_+(\boldsymbol{k})\overline{\widehat{v}_+(\boldsymbol{k})} + \widehat{u}_-(\boldsymbol{k})\overline{\widehat{v}_-(\boldsymbol{k})}], \qquad \omega = |\boldsymbol{k}|, \qquad c = 1, \tag{A.3}$$

where $\alpha > 2$ is a free parameter. To write (A.1) in the form (A.2), one substitutes the Fourier expansion of the solution $u(\boldsymbol{r}, t)$

$$u(\boldsymbol{r}, t) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{k}}{2\omega} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} [\mathrm{e}^{-\mathrm{i}\omega t}\widehat{u}_+(\boldsymbol{k}) + \mathrm{e}^{\mathrm{i}\omega t}\widehat{u}_-(\boldsymbol{k})], \tag{A.4}$$

for $\boldsymbol{r} = \boldsymbol{b}$, $t = \tau a$ into (A.1), changes the order of integration, calculates the inner integral and obtains

$$U(\boldsymbol{b}, a) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{k}}{\omega} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{b}} [\theta(-\omega a)\,\mathrm{e}^{\omega a}\widehat{u}_+(\boldsymbol{k}) + \theta(\omega a)\,\mathrm{e}^{-\omega a}\widehat{u}_-(\boldsymbol{k})]. \tag{A.5}$$

This expression has the form (A.2) if the Fourier transform of wavelets reads

$$\widehat{\Psi}^{b,a}(\boldsymbol{k}) = \omega^{\alpha-1}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{b}}[\theta(-a)\,\mathrm{e}^{\omega a} + \theta(a)\,\mathrm{e}^{-\omega a}]. \tag{A.6}$$

The inverse Fourier transform calculated in accordance with (A.4) yields

$$\Psi^{b,a}(\boldsymbol{r},t) = \frac{1}{a^4}\Psi\left(\frac{\boldsymbol{r}-\boldsymbol{b}}{a},\frac{t}{a}\right), \tag{A.7}$$

where the mother wavelet reads

$$\Psi(\boldsymbol{r},t) = \frac{\Gamma(\alpha)}{4\pi^2\mathrm{i}|\boldsymbol{r}|}\left[\frac{1}{(1-\mathrm{i}(t+|\boldsymbol{r}|))^\alpha} - \frac{1}{(1-\mathrm{i}(t-|\boldsymbol{r}|))^\alpha}\right]. \tag{A.8}$$

The approach of G Kaiser uses a very specific form of the mother wavelet, which is closely related to the analytic signal transform. Our approach can be applied for a wider class of wavelets.

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